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THE GAME OF HEX AND THE BROUWER FIXED-POINT THEOREM

DAVID GALE

1. Introduction. The application of mathematics to games of strategy is now represented by a voluminous literature. Recently there has also been some work which goes in the other direction, using known facts about games to obtain mathematical results in other areas. The present paper is in this latter spirit. Our main purpose is to show that a classical result of topology, the celebrated Brouwer Fixed-Point Theorem, is an easy consequence of the fact that Hex, a game which is probably familiar to many mathematicians, cannot end in a draw. This fact is of some practical as well as theoretical interest, for it turns out that the two-player, two-dimensional game of Hex has a natural generalization to a game of n players and n dimensions, and the proof that this game must always have a winner leads to a simple algorithm for finding approximate fixed points of continuous mappings. This latter subject is one of considerable current interest, especially in the area of mathematical economics. This paper has therefore the dual purpose of, first, showing the equivalence of the Hex and Brouwer Theorems and, second, introducing the reader to the subject of fixed-point computations.

I should say that over the years I have heard it asserted in "cocktail conversation" that the Hex and Brouwer Theorems were equivalent, and my colleague John Stallings has shown me an argument which derives the Hex Theorem from familiar topological facts which are equivalent to the Brouwer Theorem. The proof going in the other direction only occurred to me recently, but in view of its simplicity it may well be that others have been aware of it. The generalization to n dimensions may, however, be new.

In the next section we will present the relevant facts about Hex. In Section 3 we prove the equivalence of the Hex and Brouwer Theorems. The general Hex Theorem and fixed-point algorithm are presented in the final section.

2. Hex. For a brief history of the game of Hex the reader should consult [2]. The game was invented by the Danish engineer and poet Piet Hein in 1942 and rediscovered at Princeton by John Nash in 1948. It was produced commercially by Parker Brothers in 1952 but has been "out of print" in this country for many years. However, the game is still very popular and much played in other countries, including France, Germany, and Israel.

Figure 1 represents an 11×11 Hex board. The rules of the game are very simple. As in tic-tac-toe, the two players move alternately, marking any previously unmarked hexagon or *tile* with an x or o respectively. The game has been won by the x- (o-) player if he/she has succeeded in marking a *connected* set of tiles which meets the boundary regions X and X' (O and O'). (A set S of tiles is connected if any two members h and h' of S can be joined by a *path* $P = (h = h^1, h^2, \ldots, h^m = h')$ where h^i and h^{i+1} are adjacent.)

In Figure 1 neither player has won and it is the o-player's move, but the x-player has a sure win in 3 moves if he plays on the shaded tiles, as the reader will easily verify. The potentially winning connected set is indicated by the tiles marked by the underlined x's.

The appealing feature of Hex is that, unlike tic-tac-toe, it can never end in a draw, because, as explained in [2], "one player can block the other only by completing his own chain." To be a bit more precise let us state:

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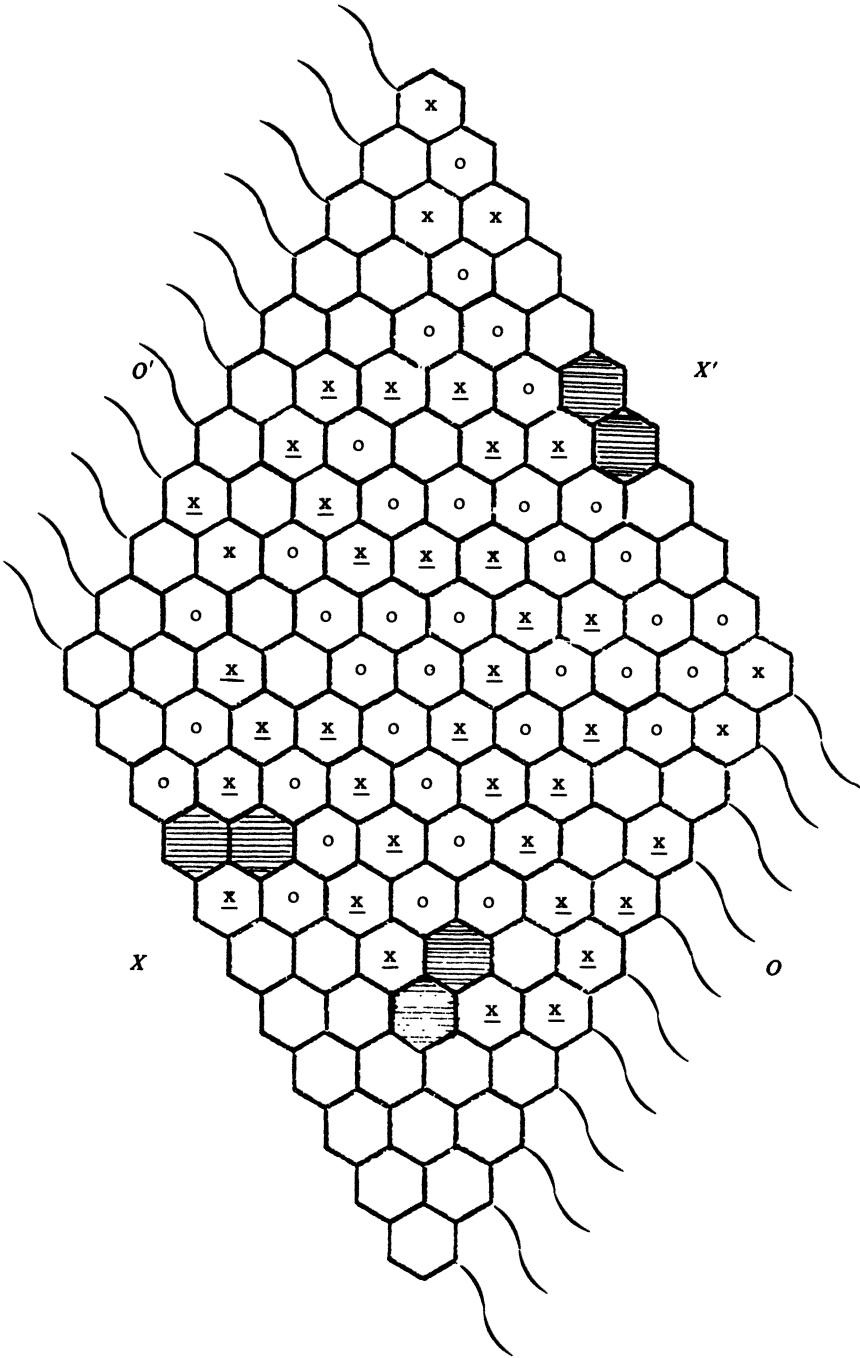
FIG. 1

HEX THEOREM. *If every tile of the Hex board is marked either* x *or* o, *then there is either an* x-*path connecting regions* X *and* X' *or an* o-*path connecting regions* O *and* O'.

The theorem is indeed intuitively quite obvious. Imagine, for example, that the X-regions are portions of opposite banks of the river "O" (as suggested by the ripples in the figure) and that the x-player is trying to build a dam by putting down stones. It is quite clear that he will have succeeded in damming the river only if he has placed his stones in a way which enables him to walk on them from one bank to the other. Nevertheless, this intuitively obvious fact is mathematically "deep," in that, as we shall see, it leads to a very short proof of the Brouwer Theorem.

Of course, intuitively obvious theorems which are hard to prove are nothing new in topology. The most celebrated case is the Jordan Curve Theorem, and it turns out that this theorem too is related to Hex; that is, one can strengthen the Hex Theorem by appending at the end of the statement the words "but not both." This latter property is, if possible, even more obvious than the fact that one of the players must win; for clearly—to continue the land-water analogy—if the x-player succeeds in constructing a causeway from X to X', he will in the process have dammed the river and prevented any flow from O to O'. The connection with the Jordan Theorem is evident. We wish to emphasize, however, that the analysis to follow and the relation to the Brouwer Theorem depends only on the no-draw property. The fact that it is not possible for both players to construct winning sets is special to two dimensions and the argument involved in proving it will not be given here, since for our purposes it is irrelevant. (Recently D. Lichtenstein has shown me a neat, purely combinatorial proof of "but not both," but the methods used—induction on the size of the board—are quite different from those to be used here to prove "at least one"; so once again it appears that these two results are mathematically quite distinct.)

How then does one prove the (weak) Hex Theorem? The only published proof I know of is in the elementary text [1]. The proof there is quite rigorous, though "informal," meaning that it uses English and pictures rather than symbols. It runs to some four pages of fine print and invokes (unnecessarily, I believe) the Jordan Curve Theorem. The proof to be presented here is closely related to that of [1] but is considerably shorter. We will also remain informal in this section so as to emphasize the intuitive geometrical ideas which also form the basis of the formal n-dimensional proof of Section 4.

Here then is our proof. Suppose the board has been covered by x's and o's as in Figure 2. By an X-face we will mean either a tile marked x or one of the regions X or X'. An O-face is defined analogously.

We consider the edge graph Γ of the Hex board to which additional edges ending in vertices u, u', v, v' have been added to separate the four boundary regions, as shown in the figure. We now present an algorithm for finding a winning set on the completely marked board. We shall make a tour along Γ, starting from the vertex u and following the simple rule of always proceeding along an edge which is the common boundary of an X-face and an O-face. Note that the edge from u has this property since it separates regions X and O. The key observation is that this touring rule determines a unique path; for suppose one has proceeded along some edge e and arrives at a vertex w. Two of the three faces incident to w are those of which e is the common boundary, hence one is an X-face, the other an O-face. The third face incident to w may be either an X-face or an O-face, but in either case there is exactly one edge e' which satisfies the touring rule. Figure 2 makes this clear (in Section 4 the situation will be completely "arithmetized" so that no picture will be needed).

Again one could make the seemingly stronger touring rule that the traveler always keep an X-face on his "right" and a O-face on his "left," but this would involve getting into the quite complex notion of orientation, which is not needed for our proof. The proof depends now on the fact that our touring rule guarantees that *we will never revisit any vertex*. On the other hand, since there are only a finite number of vertices of the graph, the tour must terminate; but the
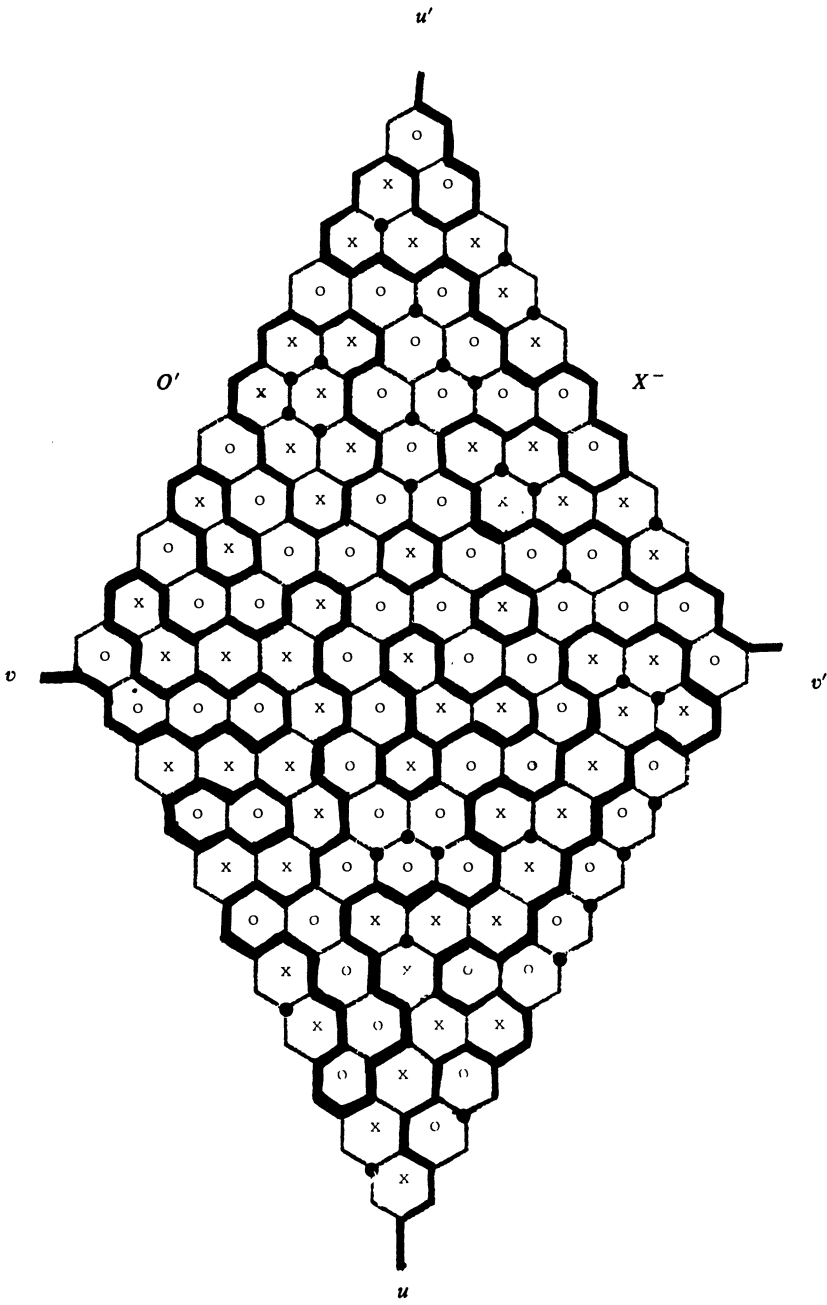
Fig. 2

only possible terminals are the vertices u', v, and v' (actually u' is not a possibility because of the "right-left" condition; but, again, we need not prove this more difficult fact). Now note that each of these three vertices is incident to one of the regions X' or O' (or both in the case of u'). If, say, the terminal vertex is incident to X', then the X-player has a winning set; for clearly the set of X-faces which are incident to our edge tour is a connected set and meets both X and X'.

Before concluding the section we wish to point out that the crucial feature of our algorithm is the italicized statement in the above paragraph which guarantees that the procedure cannot "cycle." In fact, the result which is the basis for all "fixed-point-chasing" algorithms is the following obvious fact from graph theory.

GRAPH LEMMA. *A (finite) graph whose vertices have degree at most two is the union of disjoint subgraphs, each of which is either* (i) *an isolated vertex,* (ii) *a simple cycle,* (iii) *a simple path.*

We will not insult our readers' intelligence by presenting a proof of this simple fact but suggest that they take a few minutes to see how it goes. Induction on the number of edges is recommended. The connection with the Hex algorithm should be clear. If we consider only subgraph Γ' of Γ consisting of edges which separate an X-face and an O-face, then clearly the hypothesis of the Graph Lemma is satisfied and the conclusion shows that the component of Γ starting from u must end up at one of the other degree-one vertices u', v, v'.

Figure 2 shows the graph Γ' for the completely marked board. It consists in this case of 6 simple cycles, 30 isolated vertices, and the two simple paths, in this case from u to v' and from u' to v.

3. The Equivalence of the Hex and Brouwer Theorems. For analytic purposes it is convenient to use a different but equivalent model for the Hex board. Actually, when John Nash rediscovered the game in 1948, he thought of it as being played on a checkerboard where two squares were considered to be adjacent if they were next to each other horizontally, vertically, or along a positively sloping diagonal. It is clear that this is combinatorially equivalent to the more aesthetic hexagon arrangement. This representation is easily arithmetized in a way which will generalize to n dimensions.

Notation.

Z^n denotes the lattice points of R^n. For $x \in R^n$, $|x| = \max_i x_i$. For $x \neq y \in R^n$, $x < y$ if $x_i \leqslant y_i$ for all i. The points x and y are called *comparable* if $x < y$ or $y < x$.

The (two-dimensional) *Hex board* B_k *of size* k is a graph whose *vertices* consist of all z in Z^2 with $(1,1) \leqslant z \leqslant (k,k)$. Two vertices z and z' are *adjacent* (span an edge) in B_k if

$$|z - z'| = 1, \tag{3.1}$$

$$z \text{ and } z' \text{ are comparable.} \tag{3.2}$$

Figure 3 gives the graphical representation of a Hex board of size 5. The boundary edges are labeled by the points of the compass N, S, E, W and consist of the vertices $z = (z_1, z_2)$ of B_k with $z_2 = k$, $z_2 = 0$, $z_1 = k$, $z_1 = 0$, respectively. We think of the *horizontal (vertical)* player as trying to make a path connecting E and W (N and S). We can now restate

HEX THEOREM. *Let* B_k *be covered by two sets* H *and* V. *Then either* H *contains a connected set meeting* E *and* W *or* V *contains a connected set meeting* N *and* S.

A purely combinatorial proof of this theorem generalized to n dimensions will be given in the next section. In this section we will show that it is equivalent to

BROUWER FIXED-POINT THEOREM. *Let* f *be a continuous mapping from the unit square* I^2 *into itself. Then there exists* $x \in I^2$ *such that* $f(x) = x$.

We first show, that "Hex" implies "Brouwer." Let $f : I^2 \to I^2$ be given by $f(x) = (f_1(x), f_2(x))$. From compactness of I^2 it suffices to show that for any $\varepsilon > 0$ there exists $x \in I^2$ such that
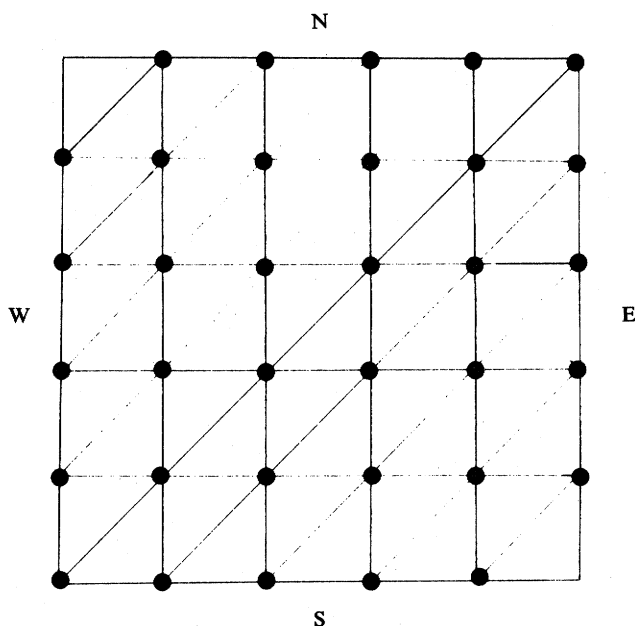
Fig. 3

$|f(x)-x|<\varepsilon$. By uniform continuity of f we know that, given $\varepsilon>0$, there is a $\delta>0$ such that $\delta<\varepsilon$ and if $|x-x'|<\delta$ then $|f(x)-f(x')|<\varepsilon$.

Now consider the Hex board B_K where $1/k<\delta$. We will define four subsets H^+, H^-, $\bar{V}^+$, V^- of B_K as follows:

$$H^+=\{z\,|\,f_1(z/k)-z_1/k>\varepsilon\}$$
$$H^-=\{z\,|\,z_1/k-f_1(z/k)>\varepsilon\}$$
$$V^+=\{z\,|\,f_2(z/k)-z_2/k>\varepsilon\}$$
$$V^-=\{z\,|\,z_2/k-f_2(z/k)>\varepsilon\}.$$

Intuitively, a vertex z belongs to H^+, H^-, V^+, V^- according as z/k is moved by f at least ε units to the right, left, up, or down, respectively.

The theorem will be proved if we can show that these four sets do not cover B_k; for, if vertex z lies in none of them, then $|f(z/k)-z/k|<\varepsilon$. The key observation is now that the (disjoint) sets H^+ and H^- (V^+ and V^{-1}) are not *contiguous* (a pair of subsets A and B of a graph are called contiguous if there exists $a\in A$ and $b\in B$ where a and b are adjacent). That is, if $z\in H^+$ and $z'\in H^-$, then

$$f_1(z/k)-z_1/k>\varepsilon$$

and

$$z_1'/k-f_1(z'/k)>\varepsilon.$$

Adding gives

$$f_1(z/k)-f_1(z'/k)+z_1'/k-z_1/k>2\varepsilon, \tag{3.3}$$

but $z_1'/k-z_1/k<\delta<\varepsilon$ by the choice of δ and k, so

$$z_1'/k-z_1'/k>-\varepsilon. \tag{3.4}$$

Adding (3.3) and (3.4) gives

$$f_1(z/k)-f_1(z'/k)>\varepsilon, \tag{3.5}$$

which shows that z and z' are not adjacent; for if they were we would have $|z/k - z'/k| = 1/k < \delta$, which would contradict the choice of δ.

Similarly V^+ and V^- are not contiguous. Now let $H = H^+ \cup H^-$, $V = V^+ \cup V^-$, and suppose Q is a connected set lying on H. From the previous paragraph, Q must lie entirely in H^+ or H^-. But note that H^+ does not meet E since f maps into I^2, and hence no point on the right boundary can be mapped to the right. Similarly, H^- does not meet W, so Q cannot meet both E and W. Similarly, V contains no connected set meeting both N and S. By the Hex Theorem, therefore, the sets H and V do not cover B_k, completing the proof. ∎

The proof that "Brouwer" implies "Hex," based on a suggestion of John Stallings modified by one of Michael Todd, makes use of the fact that the Hex board B_k gives a *triangulation* of the $k \times k$ square I_k^2 in R^2. It is easy to prove the geometrically obvious fact that each point of I^2 is uniquely expressible as a convex combination of some set of (at most three) vertices, all of which are (pairwise) adjacent. These sets of vertices are the edges and triangles of Figure 3. The reader who is not familiar with this can see how it goes by working out an algorithm which for each point $x = (x_1, x_2)$ in I_k^2 picks out the set of mutually adjacent vertices in whose convex hull x lies.

We will make use also of the fact that any mapping f from B_k into R^2 extends to a continuous *simplicial* (or *piecewise linear*) map $\hat{f}$ on I_k^2. Namely, if $x = \lambda_1 z^1 + \lambda_2 z^2 + \lambda_3 z^3$ (where the λ_i are non-negative numbers summing to 1) then by definition $\hat{f}(x) = \lambda_1 f(z^1) + \lambda_2 f(z^2) + \lambda_3 f(z^3)$.

Assume then that B_k is partitioned by two sets H and V, and again define four sets as follows: let $\hat{W}$ be all vertices connected to W by an H-path and let $\hat{E} = H - \hat{W}$. Let $\hat{S}$ be all vertices connected to S by a V-path and let $\hat{N} = V - \hat{S}$. From the definition it is clear that $\hat{W}$ and $\hat{E}$ ($\hat{N}$ and $\hat{S}$) are not contiguous. Our proof is by contradiction assuming there is no H-path from E to W and no V-path from N to S. Let e^1 and e^2 be the unit vectors of R^2 and define the mapping f from B_k into itself by

$$f(z) = z + e^1 \text{ for } z \in \hat{W}$$
$$= z - e^1 \text{ for } z \in \hat{E}$$
$$= z + e^2 \text{ for } z \in \hat{S}$$
$$= z - e^2 \text{ for } z \in \hat{N}.$$

For each of the four possibilities above one must verify that $f(z)$ is indeed in I_k^2. For example, $z + e^1$ is not in B_k only if $z \in E$; but by the assumption that there is no H-path from W to E, we see that $\hat{W}$ does not meet E. It is also true (but for a different reason) that $\hat{E}$ does not meet W. The reader should check the remaining two cases as well.

We now extend f simplicially to all of I_k^2 and obtain a contradiction by showing that f has no fixed point. This is a consequence of the following simple algebraic fact.

LEMMA 1. *Let z^1, z^2, z^3 be vertices of any triangle $\triangle$ in R^2 and let $\hat{\rho}$ be the simplicial extension of the mapping ρ defined by $\rho(z^i) = z^i + v^i$ where v^1, v^2, v^3 are given vectors. Then f has a fixed point if and only if 0 lies in the convex hull of v^1, v^2, v^3.*

∎ Let $x = \lambda_1 z^1 + \lambda_2 z^2 + \lambda_3 z^3$. Then $\hat{\rho}(x) = \lambda_1(z^1 + v^1) + \lambda_2(z^2 + v^2) + \lambda_3(z^3 + v^3)$ and x is fixed if and only if $\lambda_1 v^1 + \lambda_2 v^2 + \lambda_3 v^3 = 0$. ∎

We apply this Lemma to the mapping f. Again the key fact is the non-contiguousness of $\hat{W}$ and $\hat{E}$ ($\hat{S}$ and $\hat{N}$) which implies that if one looks at three vertices of any triangle of mutually adjacent vertices it never happens that one of these vertices is translated by e^i and another by $-e^i$. This means that the three vertices are translated by vectors which lie in a single quadrant of R^2 and hence do not have 0 in their convex hull. We have thus obtained a fixed-point-free mapping, which contradicts the Brouwer Theorem. ∎

4. The n-dimensional Hex Theorem. The formal definition of an n-dimensional Hex board is a direct generalization of the one used in Section 3 for the case $n = 2$.

DEFINITION. The *n-dimensional Hex board of size* k, H_k^n consists of all vectors *(vertices)* $z = (z_i, \ldots, z_n) \in Z^n$ such that $1 \leqslant z_i \leqslant k$, $i = 1, \ldots, n$. A pair of vertices z and z' are *adjacent* if

$$|z - z'| = 1 \tag{4.1}$$

and

$$z \text{ and } z' \text{ are } comparable \tag{4.2}$$

(meaning again $z_i \geqslant z_i'$ for all i or $z_i' \geqslant z_i$ for all i).

To avoid typographical clutter we will henceforth omit subscripts and superscripts and denote H_k^n simply by H.

For each i we define

$$H_i^- = \{z \,|\, z \in H, z_i = 1\}$$
$$H_i^+ = \{z \,|\, z \in H, z_i = k\}.$$

A *labeling* of H is a mapping L from H to $N = \{1, 2, \ldots, n\}$. We can now state for the third time

HEX THEOREM. *For any labeling L there is at least one $i \in N$ such that $L^{-1}(i)$ contains a connected set which meets H_i^- and H_i^+.* (Such a set will be called a *winning i-set*.)

We remark that the proof that the n-dimensional Hex and Brouwer Theorems are equivalent is obtained by a purely mechanical generalization of the two-dimensional proof of the previous section. From the game-theoretic point of view n-person Hex is probably not of much interest. Aside from the mechanical problem of designing a practical board there is the usual difficulty of multi-person games that players may form coalitions, and there seems to be no very satisfactory theory as to how these should be formed. It is of some interest to note that in view of the Hex theorem various variants of ordinary Hex are possible. For example, the rule could be that the first player who connects his opposite faces is the loser rather than the winner, or that player i wins if player $i + 1$ connects his opposite faces.

To prove the theorem we define the *augmented Hex board* $\hat{H}$ to be all $z \in Z^n$ with $0 \leqslant z_i \leqslant k + 1$. We further define

$$F_i^+ = \{z \,|\, z \in \hat{H}, z_i = k + 1\}$$
$$F_i^- = \{z \,|\, z \in \hat{H}, z_i = 0\}.$$

We refer to the sets F_i^+, F_i^- as the *faces* of $\hat{H}$.

For our proof we need some further terminology.

Let e^i be ith unit vector in R^n and let e be the n-vector, all of whose coordinates are 1. A *simplex* of R^n (more usually an n-simplex) is an $(n + 1)$-tuple of *vertices* $\sigma = (z^0, \ldots, z^n)$ where $z^i \in Z^n$ and

$$z^{i+1} - z^i = e^r \quad \text{for some } r \in N \tag{4.3}$$

and

$$z^{i+1} - z^i \neq z^{j+1} - z^j \quad \text{for } i \neq j. \tag{4.4}$$

Notice that for $\sigma \subset \hat{H}$ every pair of z^i and z^j in σ are adjacent from (4.1) and (4.2). The *i-facet* of σ is the n-tuple $\tau^i = (z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^n)$. An important simplex in what follows will be $\sigma^0 = (0, e^1, e^1 + e^2, \ldots, e)$. Note that all vertices of this simplex lie in H, and its n-facet $\tau^0 = (0, \ldots, e^1 + e^2 + \cdots + e^{n-1})$ lies in F_n^-.

For $0 < i < n$ the *i-neighbor* of σ is the simplex $\tilde{\sigma}$ whose vertices are the same as those of σ except that z^i is replaced by $\tilde{z}^i = z^{i-1} - z^i + z^{i+1}$. The vertex $\tilde{z}^i$ is called the *mate* of z^i (with respect to σ). The reader should verify that σ satisfies (4.3), (4.4) and also verify the *symmetry property* that $\tilde{\sigma}$ is the i-neighbor of σ if and only if σ is the i-neighbor of σ and that σ and $\tilde{\sigma}$

intersect in their common i-facet. We also define the 0-neighbor of σ to be $\tilde{\sigma} = (z^1, \ldots, z^n, \tilde{z}_0)$ where $\tilde{z}_0 = z^1 - z^0 + z^n$ and the n-neighbor of σ to be $\tilde{\sigma} = (\tilde{z}^n, z^0, \ldots, z^{n-1})$ where $\tilde{z}^n = z^{n-1} - z^n + z^0$. Again z^0 and $\tilde{z}^0$ and also z^n and $\tilde{z}^n$ are called mates, and if $\tilde{\sigma}$ is the 0-neighbor of σ then σ is the n-neighbor of $\tilde{\sigma}$, and $\sigma \cap \tilde{\sigma}$ is the n-facet of σ and the 0-facet of $\tilde{\sigma}$.

We now extend the labeling L to $\hat{H}$ by defining it on the faces of $\hat{H}$ using the *lexicographic rule*.

$$L(z) = \text{Min } \{i | z \in F_i^-\} \quad \text{if } z \in \cup F_i^-$$
$$= \text{Min } \{i | z \in F_i^+\} \quad \text{otherwise.} \tag{4.5}$$

We need one more definition. A simplex σ or facet τ is called *completely labeled*, abbreviated c.l., if L maps σ or τ onto all of N. Note that the simplex σ^0 and its n-facet τ^0 are c.l. since, from (4.5), vertices of τ^0 have labels $1, 2, \ldots, n$. We now define a graph Γ whose nodes are all c.l. simplices in $\hat{H}$. A pair of such simplices σ and $\tilde{\sigma}$ are defined to be adjacent if they are neighbors and their intersection is a c.l. facet. The crucial observation is as in Section 2.

LEMMA. *Every node σ of Γ has degree at most 2.*

■ Let $\sigma = (z^0, \ldots, z^n)$ be c.l. Then it has exactly two nodes z^i, z^j which have the same label. Thus $\tilde{\sigma}$ is a c.l. neighbor of σ if and only if it is the i or j neighbor of σ. (For any other neighbor the facet corresponding to their intersection would not be c.l.) ■

We now observe that the simplex σ^0 has exactly one c.l. neighbor. That is, suppose $L(e) = i > 1$. Now the n-neighbor of σ^0 is $\tilde{\sigma}^0 = (-e^n, 0, \ldots, e^1 + \cdots + e^{n-1})$ which is not in $\hat{H}$ since $-e^n$ is negative. The other vertex of σ^0 with label i is $e^1 + \cdots + e^{i-1}$ and its mate is $e^1 + \cdots + e^{i-2} + e^i$ which is in $\hat{H}$ so σ^0 has degree 1. (Exercise: What if $L(e) = 1$?)

Applying the Fundamental Graph Lemma of Section 2 we see that σ^0 is the initial node of a simple path $P = (\sigma^0, \sigma^1, \ldots, \sigma^m)$. We claim

LEMMA. *The c.l.-facet of σ^m lies on some face F_i^+ of $\hat{H}$.*

■ Let $\sigma^m = (z^0, \ldots, z^n)$. Since σ^m has only one neighbor it must be that for some i the mate $\tilde{z}^i$ of z^i is not in $\hat{H}$. Now for $0 < i < n$ this cannot happen, for $z^{i-1} < \tilde{z}^i < z^{i+1}$ and since z^{i-1} and z^{i+1} are in $\hat{H}$ so is $\tilde{z}^i$. Suppose that σ^m has no 0-neighbor; this means that $\tilde{z}^0 = z^1 - z^0 + z^n$ is not in $\hat{H}$. Let $z^1 - z^0 = e_r$. Then $\tilde{z}^0$ is not in $\hat{H}$ only if $z_r^n = k + 1$; but then from (4.4) we see that $z_r^i = k + 1$ for all $i > 0$, so the 0-facet of σ^m lies on F_r^+ as claimed.

It remains to consider the possibility that σ^m has no n-neighbor which would imply $\tilde{z}^n = z^0 - z^n + z^{n-1}$ is not in $\hat{H}$. If $z^n - z^{n-1} = e_r$ then this means $z_r^0 = 0$; but again by (4.4) this means that $z_r^i = 0$ for $i < n$, so the n-facet τ of σ^m lies in F_r^-. But from (4.5) τ can only have labels $i \leqslant r$, so since τ is c.l. it follows that $r = n$ and $\tau \in F_n^-$. To show that this is impossible we show that if $\tau = (z^0, \ldots, z^{n-1})$ is c.l. and lies in F_n^- then $\tau = \tau^0$. We claim first that $z^0 = 0$; for if, say, $z^0 > 0$, then the $z_r^i > 0$ for all i, and r could not be a label of τ. Likewise $z^1 = e^1$, for if z^1 were any other unit vector then no vertex of τ would get label 2. The same argument shows that $z^2 - z^1 = e^2$ and so on. On the other hand σ^m cannot have τ^0 as a facet because then P would be a cycle, whereas we know it is a path. ■

The Hex Theorem is now proved, because if σ^m has a c.l. facet on F_i^+ then there is a winning i-set. One merely chooses the vertices labeled i in each simplex of the sequence P, and note that they form a connected set. Further the vertex $e^1 + \cdots + e^{i-1}$ of σ^0 lies on F_i^- and has label i so this set meets F_1^- and F_1^+. ■

The above proof may seem a bit involved, but its important feature is that it is entirely constructive and leads to a very simple algorithm for finding a winning i-path. More important, as we shall see in a moment, it gives an effective way of finding "*almost fixed points*" of mappings.

Here is the algorithm. Given the labeling L, examine the label of the vertex e. If $L(e) = i$, bring in the mate $\bar{z}^i$ of the vertex z^i (with respect to σ^0). Call the new simplex σ^1. Now examine the label of $\bar{z}^i$. There is exactly one other vertex of σ^1 having this label. Replace it by its mate, etc. Each time a new vertex is brought into a simplex σ^k the other vertex of σ^k with the same label is dropped and replaced by its mate. As our proof shows, one eventually constructs in this way a winning i-path.

By combining the ideas of this section with those of Section 3 one is able to locate for any continuous function f of I^n into itself points which are moved by arbitrarily small amounts. Let us be somewhat more precise and say the point x in the n-cube I^n is *moved in the direction* i if $|f(x) - x| = |f_i(x) - x_i|$ (in case of ties choose the smallest subscript). To find a nearly fixed point choose a Hex board H_k^n. The larger the value of k, the better the approximation will be. The label $L(z)$ of the vertex z is then defined to be the direction in which z/k is moved under f. Of course it is not necessary to calculate the labels of all the $n! k^n$ points of H_k^n. One simply follows the Hex algorithm, calculating the label of a vertex only if it is brought into some simplex of the path P. The Hex theorem guarantees that at some point one must encounter two vertices, z and z', which are adjacent and such that both are moved in the direction i but the first is moved *forward*, that is, $f_i(z) - z_i \geqslant 0$; and the second is moved *backward*, $f_i(z) - z_i \geqslant 0$. This follows from the fact that points on the back (front) i-face of the n-cube cannot move backward (forward). But, clearly, if two nearby points are moved in opposite directions then neither can be moved very far if k is large. (The statement may be made precise by the usual ε-δ uniform continuity argument.)

For more information on the subject of fixed-point chasing the text [3] is recommended.

References

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THE "REAL" PROJECTIVE PLANE WITHOUT CONTINUITY

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The title of this article refers to the elegant treatise *The Real Projective Plane* by H. S. M. Coxeter [1], in which the main theorems of classical plane projective geometry are proved synthetically from axioms of incidence and order. The only non-elementary axiom is the axiom

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